International Journal of Theoretical Physics, Vol. 44, No. 8, August 2005 (© 2005)
DOI: 10.1007/s10773-005-4685-x

Hamilton–Jacobi Quantization of Singular Lagrangians with Linear Velocities

Sami I. Muslih,[1,2,5] Hosam A. El-Zalan,[3] and Eqab M. Rabei[4]

Received July 21, 2004; accepted August 25, 2004

In this paper, constrained Hamiltonian systems with linear velocities are investigated by using the Hamilton–Jacobi method. The integrablity conditions are considered on the equations of motion and the action function as well in order to obtain the path integral quantization of singular Lagrangians with linear velocities.

KEY WORDS: Hamiltonian and Lagrangian approaches; Hamilton–Jacobi approach; total differential equations; integrable systems.

1. INTRODUCTION

The study of singular Lagrangian with linear velocity has been dealt within the last 50 years by Dirac's Hamiltonian formulism (Dirac, 1964, 1950). In this formalism Dirac showed that the algebra of Poisson brackets determines a division of constraints into two classes: The so-called the first-class and the second-class constraints. The first-class constraints are those that have zero Poisson brackets with all other constraints in the subspace of phase space in which constraints hold; constraints which are not first-class are by definition second-class. Also in his method, the Poisson brackets in a second class constraints systems are converted into Dirac brackets to attain self-consistency. However, wherever we adopt the Dirac method, we frequently meet the problem of the operating ordering ambiguity. Besides, the presence of first class constraints in such theories requires care when applying Dirac's method, since the first class constraints are the generators of gauge transformations which lead to the gauge freedom. In other words, the equations of motion are still degenerate and depend on the functional

[1] International Center for Theoretical Physics (ICTP), Trieste, Italy.
[2] Department of Physics, Al-Azhar University-Gaza, Palestine.
[3] Department of Mathematics, Al-Aqsa University-Gaza, Palestine.
[4] Department of Physics, Mu'tah University, Al-Karak, Jordan.
[5] To whom correspondence should be addressed at Al-Azhr University-Gaza, Palestine; e-mail: smuslih@ictp.trieste.it.

arbitraries, one has to impose external gauge fixing constraint for each first class constraint which is not always an easy task.

Recently, Güler (1987, 1989, 1992a,b) have proposed an alternating approach to constrained systems that avoids the separations of constraints into first and second class and the use of weak and strong equations. This new method of analysis has been successfully used by many authors (Muslih and Güler, 1995, 1998; Rabei and Güler, 1992; Muslih, 2003a; Muslih *et al.*, 2003) and is by now a standard technique to deal with constrained system. Besides, the canonical path integral method based on the Hamilton–Jacobi method have been initiated in (Muslih, 2000a,b,c,d, 2002a,b, 2003b, 2004a; Muslih *et al.*, 2000) to obtain that path integral quantization as an integration over the canonical phase space coordinate without any need to use any gauge fixing conditions (Muslih and Güler, 1995, 1998; Rabei and Güler, 1992) as well as, no need to enlarge the initial phase-space by introducing unphysical auxiliary fields (Muslih, 2004b,c).

Some authors (Rabei *et al.*, 2003; Baleanu and Güler, 2000; Corichi, 1992; Hojman and Shepley, 1990; Kulshreshtha, 1992) have investigated singular Lagrangians with linear velocities using Dirac's procedure. For example, using the Lagrangian given in reference (Hojman and Shepley, 1990), there are different approaches that give different results. Besides, in Baleanu and Güler (2000) the authors have investigated singular Lagrangian with linear velocities without considering the integrability condition on the action function. On the other hand in reference (Rabei *et al.*, 2003), the authors have investigated singular Lagrangian with linear velocities by using the Hamilton–Jacobi method and obtained the integrable action directly without considering the total variation of constraints.

In this paper, we shall consider integrability condition on the equations of motions and the action function as well in order to obtain the path integral quantization of singular Lagrangian with linear velocities.

2. THE HAMILTON–JACOBI METHOD

In this section, we shall briefly review the Hamiltonian formulation of constrained systems (Güler, 1987, 1989, 1992a,b). The starting point of this method is to consider the Lagrangian $L = L(q_i, \dot{q}_i, t),\ i = 1, \ldots, n$, with the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i,\ j = 1, \ldots, n, \tag{1}$$

of rank $(n - r)$, $r < n$. Then r momenta are dependent. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \ldots, n - r, \tag{2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}, \quad \mu = n - r + 1, \ldots, n, \tag{3}$$

where q_i are divided into two sets, q_a and x_μ. Since the rank of the Hessian matrix is $(n - r)$, one can solve Eq. (2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{x}_\mu, p_a; t). \tag{4}$$

Substituting Eq. (4) into Eq. (3), we get

$$p_\mu = -H_\mu(q_i, \dot{x}_\mu, p_a; t). \tag{5}$$

The canonical Hamiltonian H_0 reads

$$H_0 = p_a \dot{q}_a + p_\mu \dot{x_\mu}|_{p_\nu = -H_\nu} - L(t, q_i, \dot{x_\nu}, \dot{q_a}), \quad \mu,\ \nu = n - r + 1, \ldots, n. \tag{6}$$

The set of Hamilton–Jacobi partial differential equations (HJPDE) is expressed as (Güler, 1987, 1989, 1992a,b)

$$H_\alpha^{'}\left(x_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial x_\alpha}\right) = 0, \quad \alpha,\ \beta = 0, n - r + 1, \ldots, n, \tag{7}$$

where

$$H'_0 = p_0 + H_0, \quad p_0 = \frac{\partial S}{\partial t} \tag{8}$$

$$H_\mu^{'} = p_\mu + H_\mu, \quad p_\mu = \frac{\partial S}{\partial x_\mu} \tag{9}$$

we define $p_\beta = \partial S[q_a; x_\alpha]/\partial x_\beta$ and $p_a = \partial S[q_a; x_\alpha]/\partial q_a$ with $x_0 = t$ and S being the action. The equations of motion are obtained as total differential equations in many variables as follows (Güler, 1987, 1989, 1992a,b):

$$dq_a = \frac{\partial H_\alpha^{'}}{\partial p_a} dx_\alpha, \qquad dp_a = -\frac{\partial H_\alpha^{'}}{\partial q_a} dx_\alpha, \tag{10}$$

$$dp_\beta = -\frac{\partial H_\alpha^{'}}{\partial x_\beta} dx_\alpha, \tag{11}$$

$$dz = \left(-H_\alpha + p_a \frac{\partial H_\alpha^{'}}{\partial p_a}\right) dx_\alpha, \tag{12}$$

where $z = S(x_\alpha; q_a)$. The analysis of a constrained system is reduced to solve equations (10) with constraints $H_\alpha^{'} = 0$. Variation of constraints (7) considering equations (10) may vanish identically or give rise to new constraints. In the case of new constraints we should consider their variations also. Repeating this procedure, one may obtain a set of constraints such that all the variations vanish. Simultaneous solutions of canonical equations with all these constraints provide the solution of a singular system. In fact, in references (Muslih and Güler, 1995, 1998; Rabei and Güler, 1992), the integrability conditions for Eqs. (10) and (11) are discussed without considering the integrability conditions of the action function.

3. INTEGRATION OF TOTAL DIFFERENTIAL EQUATIONS

As was clarified, that Eqs. (10)–(12) are obtained as total differential equations in many variables, which require the investigation of integrability conditions. To achieve this goal, we define the linear operators X_α that correspond to the total differential Eqs. (6) and (7) as

$$\begin{aligned} X_\alpha f(t_\beta, q_a, p_a, z) &= \frac{\partial f}{\partial t_\alpha} + \frac{\partial H'_\alpha}{\partial p_a}\frac{\partial f}{\partial q_a} - \frac{\partial H'_\alpha}{\partial q_a}\frac{\partial f}{\partial p_a} \\ &\quad + (-H_\alpha + p_a\frac{\partial H'_\alpha}{\partial p_a})\frac{\partial f}{\partial z}, \\ &= [H'_\alpha, f] - \frac{\partial f}{\partial z}H'_\alpha, \qquad (13) \\ &\quad \alpha, \beta = 0, n-r+1, \ldots, n, a = 1, \ldots, n-r, \end{aligned}$$

where the commutator [,] is the square bracket (for details, see Section 7).

Lemma. A system of total differential equations (10)–(12) is integrable if and only if

$$\{H'_\alpha, H'_\beta\} = 0, \quad \forall\, \alpha,\ \beta, \qquad (14)$$

where the commutator {, } is the Poisson bracket (for details see references Muslih, 2003a, 2004a,b,c).

Equation (14) are the necessary and sufficient conditions that the system (10)–(12) of total differential equations be integrable.

If the set of Eqs. (10)–(12) is integrable, then one can obtain the canonical action function (12) in terms of the canonical coordinates. In this case, the path integral representation may be written as (Muslih, 2000a,b,c,d, 2002a,b, 2003b, 2004a; Muslih *et al.*, 2000)

$$\begin{aligned} \Psi(q'_a, t'_\alpha; q_a, t_\alpha) &= \int_{q_a}^{q'_a} Dq^a\, Dp^a \\ &\quad \times \exp i\left\{\int_{t_\alpha}^{t'_\alpha}\left[-H_\alpha + p_a\frac{\partial H'_\alpha}{\partial p_a}\right]dt_\alpha\right\}, \\ a &= 1, \ldots, n-r, \quad \alpha = 0, n-r+1, \ldots, n. \qquad (15) \end{aligned}$$

4. THE MODEL

In this section, we would like to investigate singular systems with linear Lagrangians using the Hamilton–Jacobi method (Güler, 1987, 1989, 1992a,b; Muslih, 2003b, 2003c), in order to obtain the path integral quantization for these systems. Let us consider the following linear Lagrangian (Rabei *et al.*, 2003; Baleanu and Güler, 2000)

$$L = a_i(q_j)\dot{q}_i - V(q_j), \tag{16}$$

where $a_i(q)$ and $V(q)$ are continuous functions of q_j.

The generalized canonical momentum corresponding to this Lagrangian are given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = a_i(q) = -H_i. \tag{17}$$

The primary constraints are

$$H_i^{'} = p_i - a_i. \tag{18}$$

The canonical Hamiltonian H_0 is given by:

$$H_0 = p_i\dot{q}_i - L = V(q). \tag{19}$$

The corresponding HJPDEs are

$$H_0^{'} = p_0 + H_0 = p_0 + V(q) = 0, \tag{20}$$

$$H_i^{'} = p_i + H_i = p_i - a_i(q) = 0. \tag{21}$$

The equations of motion are obtained as total differential equations follows:

$$dq_i = \frac{\partial H_0^{'}}{\partial p_i}dt + \frac{\partial H_j^{'}}{\partial p_i}dq_j = dq_i, \tag{22}$$

$$dp_i = -\frac{\partial H_0^{'}}{\partial q_i}dt - \frac{\partial H_j^{'}}{\partial q_i}dq_j = -\frac{\partial V(q)}{\partial q_i}dt + \frac{\partial a_j(q)}{\partial q_i}dq_j. \tag{23}$$

To check whether the set of Eqs. (22) and (23) are integrable or not, let us consider the total variation of (21). In fact

$$\begin{aligned} dH_i^{'} &= dp_i - da_i(q) = 0 \\ &= -\frac{\partial V(q)}{\partial q_i}dt + \frac{\partial a_j(q)}{\partial q_i}dq_j - da_i(q) \end{aligned} \tag{24}$$

So, we have

$$\frac{\partial a_i(q)}{\partial q_j}dq_j - \frac{\partial a_j(q)}{\partial q_i}dq_j = -\frac{\partial V(q)}{\partial q_i}dt, \tag{25}$$

or

$$\dot{q}_j = -f_{ij}^{-1}\,\frac{\partial V(q)}{\partial q_i}, \tag{26}$$

where the anti-symmetric matrix f_{ij} is given by

$$f_{ij} = \frac{\partial a_i(q)}{\partial q_j} - \frac{\partial a_j(q)}{\partial q_i}. \tag{27}$$

Making use of (12), (20) and (21), we can write the canonical action integral as

$$S = \int a_i dq_i - \int V(q))\,dt. \tag{28}$$

In order to obtain the path integral quantization for the model (16), we follow the Faddeev, Jackiw method (Jackiw, 1993) for the case that the inverse of the matrix f_{ij} exists. In this case the path integral Ψ is given by

$$\begin{aligned}\Psi &= \int \prod_{i=1}^{n} Dq_i\; det^{\frac{1}{2}} f_{jk}\; e^{iS},\\ &= \int \prod_{i=1}^{n} Dq_i\; det^{\frac{1}{2}} f_{jk}\; e^{i\{\int a_i dq_i - \int V(q)dt\}}.\end{aligned} \tag{29}$$

The integrable action can be obtained using the Hamilton–Jacobi method. In fact

$$\int d(a_i q_i) = a_i q_i = \int a_i dq_i + \int q_i da_i. \tag{30}$$

So

$$S = \frac{1}{2} a_i q_i - \frac{1}{2} \int \left[q_j da_i - a_i dq_i + 2V(q)\,dt \right]. \tag{31}$$

Making use of (24) and (30), the action function becomes

$$S = \frac{1}{2} a_i q_i - \frac{1}{2} \int \left[\left(2V - q_j \frac{\partial V}{\partial q_i} \right) dt + \left(\frac{\partial a_j}{\partial q_i} q_j - a_i \right) dq_i \right]. \tag{32}$$

Now, assuming that the functions $a_i(q_j)$ and $V(q_j)$ satisfy the following conditions

$$2V = \frac{\partial V}{\partial q_i} q_j, \qquad a_i = \frac{\partial a_j}{\partial q_i} q_j,$$

we obtain the integrable action as follows:

$$S = \frac{1}{2} a_i q_i + c, \tag{33}$$

where c is some constant.

Hence, the path integral quantization (29) can be put in the form

$$\Psi = \int \prod_{i=1}^{n} Dq_i \; det^{\frac{1}{2}} f_{jk} \; e^{i(\frac{1}{2}a_i q_i)}. \tag{34}$$

5. AN EXAMPLE

As an example, let us consider the following linear (singular) Lagrangian (Rabei *et al.*, 2003)

$$L = (q_2 + q_3)\dot{q}_1 + q_4\dot{q}_3 - V(q), \tag{35}$$

where the potential $V(q)$ is given by

$$V(q) = -\frac{1}{2}\left(q_4^2 - 2q_2q_3 - q_3^2\right). \tag{36}$$

The functions $a_i (i = 1, 2, 3, 4)$ are

$$a_1 = q_2 + q_3, \quad a_2 = 0, \quad a_3 = q_4, \quad a_4 = 0. \tag{37}$$

Using (17), the generalized momenta corresponding to this Lagrangian are:

$$p_1 = a_1(q) = q_2 + q_3 = -H_1,$$
$$p_2 = a_2 = 0 = -H_2,$$
$$p_3 = a_3 = q_4 = -H_3,$$
$$p_4 = a_4 = 0 = -H_4.$$

The primary constraints are given as

$$H_1^{'} = p_1 - q_2 - q_3, \quad H_2^{'} = p_2, \quad H_3^{'} = p_3 + q_4, \quad H_4^{'} = p_4. \tag{38}$$

Equation (19) gives the canonical Hamiltonian H_0 as

$$H_0 = V(q) = -\frac{1}{2}\left(q_4^2 - 2q_2q_3 - q_3^2\right). \tag{39}$$

Now making use of (23), the equations of motion read as

$$dp_1 = 0, \tag{40}$$

$$dp_2 = -q_3\,dt + dq_1, \tag{41}$$

$$dp_3 = -(q_2 + q_3)\,dt + dq_1, \tag{42}$$

$$dp_4 = q_4\,dt + dq_3. \tag{43}$$

The matrix f_{ij} defined in (27) is given by

$$f_{ij} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{44}$$

The S-action function is calculated as

$$S = \frac{1}{2}\left[(q_2 + q_3)q_1 + q_4 q_3\right] + c. \tag{45}$$

Making use of (34) and (45), the path integral for the model is calculated as

$$\Psi = \int dq_1 dq_2 dq_3 dq_4 e^{i\frac{1}{2}[(q_2+q_3)q_1+q_4q_3]}. \tag{46}$$

6. CONCLUSION

In this work, we have investigated constrained systems using the Hamilton–Jacobi method for Lagrangians with linear velocities. The equations of motion are obtained from the integrability conditions. Besides the integrable action (33) is obtained from the integrability conditions, which leads us to obtain the path integral quantization (34) by using the Faddeev, Jackiw method (Jackiw, 1993) directly as an integration over the independent dynamical variables.

7. SQUARE BRACKETS AND POISSON BRACKETS

In this appendix, we shall give a brief review on two kinds of commutators: the square and the Poisson brackets.

The square bracket is defined as

$$[F, G]_{q_i, p_i, z} = \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial p_i}\left(p_i \frac{\partial G}{\partial z}\right) - \frac{\partial G}{\partial p_i}\left(p_i \frac{\partial F}{\partial z}\right). \tag{47}$$

The Poisson bracket is defined as

$$\{f, g\}_{q_i, p_i} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}. \tag{48}$$

According to above definitions, the following relation holds

$$[H_\alpha^{'}, H_\beta^{'}] = \{H_\alpha^{'}, H_\beta^{'}\}. \tag{49}$$

REFERENCES

Baleanu, D. and Güler, Y. (2000). *Nuovo Cimento B* **115**, 319.
Corichi, A. (1992). *Journal of Mathematical Physics* **33**, 4066.

Dirac, P. A. M. (1950). *Canadian Journal of Mathematics*, **2**, 129.
Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, Academic Press, New York.
Güler, Y. (1987). *Nuovo Cimento B* **100**, 251.
Güler, Y. (1989). *Journal of Mathematical Physics* **30**, 785.
Güler, Y. (1992a). *Nuovo Cimento B* **107**, 1389.
Güler, Y. (1992b). *Nuovo Cimento B* **107**, 1143.
Hojman, S. A. and Shepley, L. C. (1990). *Journal of Mathematical Physics* **32**, 142.
Jackiw, R. (1993). *(Constrained) Quantization Without Tears*, MIT CTP No. 2215 (hep-th/9306075).
Kulshreshtha, V. (1992). *Journal of Mathematical Physics* **33**, 633.
Muslih, S. I. (2000a). *Nuovo Cimento B* **115**, 1.
Muslih, S. I. (2000b). *Nuovo Cimento B* **115**, 7.
Muslih, S. I. (2000c). *Hadronic Journal* **23**, 203.
Muslih, S. I. (2000d). *Hadronic Journal Supplement* **15**, 461.
Muslih, S. I. (2002a). *General Relativity and Gravitation* **34**, 1059.
Muslih, S. I. (2002b). *Modern Physics Letters A* **36**, 2382.
Muslih, S. I. (2003a). *Nuovo Cimento B* **118**, 505.
Muslih, S. I. (2003b). *Modern Physics Letters A* **18**, 1187.
Muslih, S. I. (2004a). *On the path integral of constrained systems*. In *Proccedings of the International Workshop on Global Analysis*, 15–17 April, 2004, Cankaya University, Ankara, Turkey. (AIP Conference Proccedings Volume 729, pp. 240–247).
Muslih, S. I. (2004b). *Modern Physics Letters A* **19**, 151.
Muslih, S. I. (2004c). *Modern Physics Letters A* **19**(11), 863.
Muslih, S. I. and Güler, Y. (1995). *Nuovo Cimento B* **110**, 307.
Muslih, S. I. and Güler, Y. (1998). *Nuovo Cimento B* **113**, 277.
Muslih, S. I., El-Zalan, H. A., and El-Sabaa, F. (2000). *International Journal of Theoretical Physics* **39**, 2495.
Rabei, E. M. *et al*. (2003). *Modern Physics Letters A* **18**, 1591.
Rabei, E. M. and Güler, Y. (1992). *Physical Review A* **46**, 3513.